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NON-LINEAR OSCILLATIONS OF A HAMILTONIAN SYSTEM WITH ONE DEGREE OF FREEDOM AND FOURTH-ORDER RESONANCE†

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Non-linear oscillations of a 2π-periodic Hamiltonian system with one degree of freedom are considered. It is assumed that the origin of coordinates is an equilibrium position, the linearized system is assumed to be stable, its characteristic exponents $\pm i\nu$ are pure imaginary, and the value of 4ν is close to an integer. When the methods of classical perturbation theory are used, the investigation reduces to an analysis of a model system which can be described by the typical Hamiltonian of problems on the motion of Hamiltonian systems with one degree of freedom in the case of fourth-order resonance. The system is analysed in detail. The results for the model system are applied to the total system using Poincaré's theory of periodic motion and the KAM-theory. The existence, number and stability of 8π-periodic motions of the initial system are investigated. Trajectories of motion which start in a fairly small neighbourhood of the origin of coordinates are bounded. An estimate of the size of that neighbourhood is given. The examples considered are of a point mass above a curve in the shape of an ellipse which collides with the curve, and plane non-linear oscillations of a satellite in an elliptical orbit in the case of fourth-order resonance. © 1999 Elsevier Science Ltd. All rights reserved.

1. STATEMENT OF THE PROBLEM. TRANSFORMATION OF THE HAMILTONIAN

We will consider the motion of a system with one degree of freedom described by the Hamiltonian function $H(x, y, t)$ (x is the coordinate and y is the momentum). Let the origin of coordinates $x = y = 0$ be an equilibrium position of the system, and let the function H be analytic in the neighbourhood of $x = y = 0$ and 2π-periodic with respect to t. We will write H in the form

$$H = H_2 + H_3 + H_4 + \dots \tag{1.1}$$

where $H_k(x, y, t)$ is a kth degree polynomial in x and y.

Suppose the corresponding linearized system is Lyapunov stable and its characteristic exponents $\pm i\nu$ are pure imaginary. We shall assume that the system has no resonances up to the third order inclusive, so that the numbers 2ν and 3ν are not integers.

If there is fourth-order resonance (if 4ν is an integer), the non-linear terms in the equations of motion will either preserve the stability of the point $x = y = 0$ or destroy it [1].

Our aim here is to investigate non-linear oscillations of the system in the near-resonance case, when the number 4ν is close to the integer N. We examine the existence, number and stability of 8π-periodic motions of the system and we show that its trajectories of motion, starting in a fairly small neighbourhood of the origin, are bounded.

We first make a number of canonical replacements of variables, transforming the Hamiltonian (1.1) to a form that is typical for the resonance case considered here [1]. First, using the real replacement $x, y \to q, p$, which is a 2π-periodic with respect to t, we can reduce the function H_2 to normal form $\nu(q^2 + p^2)/2$, discard third-degree terms, and simplify the set of fourth-degree terms by leaving only resonance terms in it. Assuming then that $q = \varepsilon\sqrt{(2R)} \sin\theta, p = \varepsilon\sqrt{(2R)}\cos\theta$ ($0 < \varepsilon \ll 1$), we will write the Hamiltonian, normalized up to fourth-degree terms inclusive, in the form

$$K = \nu R + \varepsilon^2[c + a\sin(4\theta - Nt) + b\cos(4\theta - Nt)]R^2 + O(\varepsilon^3) \tag{1.2}$$

where a, b and c are constants. We shall assume that $c \neq 0$ and $a^2 + b^2 \neq 0$.

Let $4\nu = N + 4\varepsilon^2\chi$. We will make the replacement of variables $\theta, R \to \varphi, r$ using the formulae

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$$\theta = Nt/4 + \theta_* + \sigma[(1+\sigma)\pi/8 + \varphi], \quad R = \delta r$$

$$\sigma = \operatorname{sign} c, \quad \delta = \left(a^2 + b^2\right)^{-1/2}, \quad \sin 4\theta_* = a\delta, \quad \cos 4\theta_* = b\delta$$

and introduce the new independent variable $\tau = \varepsilon^2 t$. The final form of the Hamiltonian function will be

$$\Gamma = \gamma_0(\varphi, r) + \varepsilon\gamma_1(\varphi, r, \tau, \varepsilon) \tag{1.3}$$

where

$$\gamma_0 = \mu r + (\varkappa - \cos 4\varphi) r^2, \quad \mu = \sigma\chi, \quad \varkappa = |c|\delta \tag{1.4}$$

The function γ_1 is a 2π-periodic in φ and $8\pi\varepsilon^2$-periodic in τ, and in the region $0 < r \ll 1$ it is analytic with respect to all its variables.

We then assume that $\varkappa \neq 1$. The critical case $\varkappa = 1$ with exact resonance ($\mu = 0$) was studied in [2].

2. PHASE PORTRAITS OF THE MODEL SYSTEM

We will first consider the motions of a system with truncated (model) Hamiltonian (1.4), obtained from the total Hamiltonian (1.3) by discarding terms of order ε and above. The equations of motion corresponding to (1.4) have the form

$$dr/d\tau = -4r^2 \sin 4\varphi, \quad d\varphi/d\tau = \mu + 2r(\varkappa - \cos 4\varphi) \tag{2.1}$$

and the relation

$$\gamma_0(\varphi, r) = h = \text{const} \tag{2.2}$$

is their first integral.

We will indicate the equilibrium positions of system (2.1) and what type of stability they exhibit. The equilibrium position $r = 0$ exists for any values of the parameters κ and μ; if $\mu = 0$, it is stable for $\varkappa > 1$ and unstable for $0 < \varkappa < 1$, but if $\mu \neq 0$, it is a stable equilibrium position for any values of $(0 < \varepsilon \ll 1)$.

The other equilibrium positions $r = r_*$, $\varphi = \varphi_*$ of system (2.1) are found from the relations

$$\sin 4\varphi_* = 0, \quad \mu + 2r_*(\varkappa - \cos 4\varphi_*) = 0 \tag{2.3}$$

Equation (2.3) has no solutions in the region $\varkappa > 1$, $\mu > 0$.

In the regions $0 < \varkappa < 1$, $\mu > 0$ and $0 < \varkappa < 1$, $\mu < 0$, system (2.3) has four solutions: in the former, its solutions are $r_* = \mu/2(1 - \varkappa))$, $\varphi_* = 0, \pi/2, \pi, 3\pi/2$, and in the latter, $-r_* = |\mu|/(2(1 + \varkappa))$, $\varphi_* = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$. The corresponding equilibrium positions of system (2.1) are unstable.

In the region $\varkappa > 1$, $\mu < 0$, system (2.1) has eight equilibrium positions $r_* = |\mu|1/2(\varkappa - 1))$, $\varphi_* = 0, \pi/2, \pi, 3\pi/2$, and $r_* = |\mu|2(\varkappa + 1))$, $\varphi_* = 0, \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ of which the first four are stable, and the rest unstable.

The phase portraits of system (2.1) in the plane of variables $u = \sqrt{2r}\cos\varphi$, $v = \sqrt{2r}\sin\varphi$ are shown for $0 < \varkappa < 1$, $\mu = 0$ (a); $\varkappa > 1$, $\mu = 0$ (b); $0 < \varkappa < 1$, $\mu > 0$ (c); $\varkappa > 1$, $\mu > 0$ (d); $0 < \varkappa < 1$, $\mu < 0$ (e); $\varkappa > 1$, $\mu < 0$ (f) in Fig. 1. They are symmetrical about the coordinate axes and their bisectors.

When $\mu = 0$, the origin of coordinates, which is a complex singular point of the system, is stable when $\varkappa > 1$ and unstable when $0 < \varkappa < 1$.

The stable equilibrium positions of system (2.1) for $\mu \neq 0$ in Fig. 1 are represented by "central" singular points and the unstable positions by saddle points. The unstable singular points of the system are connected by separatrices which separate the regions in which the trajectories of the system behave differently.

It will be assumed below that $\mu \neq 0$. We will describe the motion of the model system for all possible values of the parameters μ, $\varkappa$ and h, using the notation $h_\pm = \mu^2/(4(1 \pm \varkappa))$.

Suppose $0 < \varkappa < 1$, $\mu > 0$ (Fig. 1c). The value $h = 0$ of the "energy" constant corresponds to a stable equilibrium position, the origin of coordinates, and the value $h = h_-$ corresponds to unstable equilibrium positions and the separatrices. Taking μ as an arbitrary fixed quantity ($\mu > 0$) we pick out three regions with boundaries $(\varkappa, h)$ and $h = 0$ in the $(\varkappa, h)$ and $h = h_-$ plane (Fig. 2a). The curve $h = h$ in Fig. 2(a)

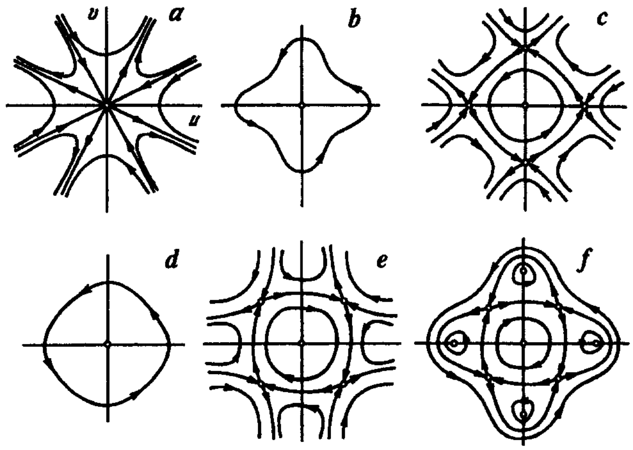

Fig. 1.

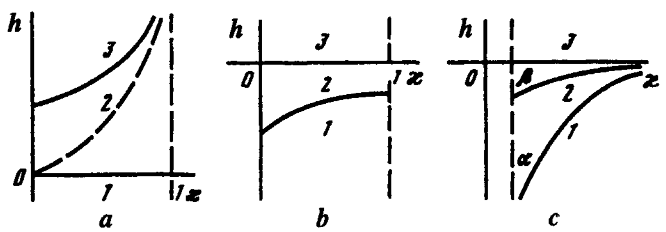

Fig. 2.

is depicted by the solid line. If $h < h_-$ (in Fig. 2a, regions 1 and 2 and the part of the straight line $h = 0$ for which $0 < \varkappa < 1$), four trajectories correspond to each value of the parameters $\varkappa$ and h in Fig. 2c—unclosed curves intersecting the coordinate axes. In addition, one of the oscillations in the neighbourhood of the origin of coordinates corresponds to each point of region 2 ($0 < h < h_-$). (For $h > h_-$ (region 3 in Fig. 2a) four unclosed trajectories intersecting the bisectors of the angles between the axes correspond to each value of $\varkappa$ and h

If $0 < \varkappa < 1$, $\mu < 0$ (Fig. 1e), then, similarly, we can divide the $(\varkappa, h)$ plane into three regions (Fig. 2*b*) with boundaries $h = 0$ and $h = -h_+$, corresponding to a stable equilibrium position—the origin of coordinates, and unstable equilibrium positions; motion along the separatrices also corresponds to points of the curve $h = -h_+$. In region 1 ($h < -h_+$) we have motion along unclosed curves intersecting the coordinate axes and for $h > -h_+$ (regions 2, 3 and the part of the straight line $h = 0$ for which $0 < \varkappa < 1$), we have motion along unclosed curves intersecting the bisectors of the angles between the axes; moreover, an oscillation in the neighbourhood of the origin of coordinates corresponds to each point of region 2 ($-h_+ < h < 0$).

If $\varkappa > 1$, $\mu < 0$ (Fig. 1f), the boundaries of the regions where system (2.1) behaves differently in the plane of parameters $\varkappa$, h are the straight line $h = 0$, corresponding to the origin of coordinates—a stable equilibrium position, and the curves $h = h_-$ and $h = -h_+$, corresponding to stable ($r_* = |\mu|/(2(\varkappa - 1))$ and unstable ($r_* = |\mu|/(2(\varkappa + 1))$ equilibrium positions of the system (curves α and β respectively in Fig. 2*c*). The points on curve β also correspond to motion along the separatrices. For $h < h_-$ (region 1 in Fig. 2*c*) motion is impossible. Four oscillation modes of the system in the neighbourhood of the stable equilibrium positions for which $r_* = |\mu|/(2(\varkappa - 1))$ correspond to each point of region 2 ($h_- < h < -h_+$) in Fig. 2c. For values of $\varkappa$ and h in region 3 ($-h_+ < h < 0$) we have either an oscillation near the origin of one of the rotations—the closed trajectories in Fig. 1f, which include all the singular points of the system. Apart from a stable equilibrium—the origin of coordinates—the value $h = 0$ corresponds to one of the rotations. One rotation corresponds to each point of region 4 ($h > 0$) in Fig. 2.

Finally, if $\varkappa > 1$, $\mu > 0$ (Fig. 1d) for $h < 0$ motion is impossible, for $h = 0$ we have stable equilibrium—the origin of coordinates, and for $h > 0$ there are oscillations in the neighbourhood of this equilibrium.

3. ANALYSIS OF THE MODEL SYSTEM

Integration. Using integral (2.2), we eliminate φ from the first equation of system (2.1). The equation for r will have the form

$$dr/\sqrt{F(r)} = \mp 4d\tau, \quad F(r) = r^4 - \left(\mu r + \varkappa r^2 - h\right)^2 \tag{3.1}$$

The upper and lower signs in (3.1) correspond to motion in the sectors $\pi k/2 < \varphi < \pi(2k+1)/4$ and $\pi(2k+1)/4 < \varphi < \pi(k+1)/2$ ($k = 0, 1, 2, 3$) of decrease and increase of the variable r respectively. The roots of the polynomial $F(r)$ have the form

$$r_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4h(1-\varkappa)}}{2(1-\varkappa)}, \quad r_{3,4} = \frac{-\mu \pm \sqrt{\mu^2 + 4h(1+\varkappa)}}{2(1+\varkappa)} \tag{3.2}$$

We will give the results of integrating Eq. (3.1) on all bounded trajectories of system (2.1). Knowing $r(\tau)$, the dependence $\varphi(t)$ on these trajectories can be obtained from relation (2.2).

In regions of oscillations of the system in the neighbourhood of the origin of coordinates (Fig. 1c–f) and in the region of rotations for $\varkappa > 1$, $\mu < 0$ (Fig. 1f), all the roots of the polynomial $F(r)$ are real.

If $0 < \varkappa < 1$, $\mu > 0$ (Fig. 1c), in the region of oscillations ($0 < h < h_-$) we have $r_4 < r_3 \leqslant r \leqslant r_2 < r_1$. Putting $r(0) = r_3$, from (3.1) we have [3]

$$r(\tau) = \frac{r_3(r_2 - r_4) - r_4(r_2 - r_3)\operatorname{sn}^2 u}{(r_2 - r_4) - (r_2 - r_3)\operatorname{sn}^2 u} \tag{3.3}$$

If $0 < \varkappa < 1$, $\mu < 0$ (Fig. 1e), in the region of oscillations ($-h_+ < h < 0$) we have $r_2 < r_1 \leqslant r \leqslant r_4 < r_3$ and from (3.1) we obtain

$$r(\tau) = \frac{r_1(r_4 - r_2) - r_2(r_4 - r_1)\operatorname{sn}^2 u}{(r_4 - r_2) - (r_4 - r_1)\operatorname{sn}^2 u}, \quad r(0) = r_1 \tag{3.4}$$

In (3.3) and (3.4)

$$u = 2\left[\left(1 - \varkappa^2\right)(r_3 - r_1)(r_4 - r_2)\right]^{1/2}\tau \tag{3.5}$$

and the modulus k of the elliptic sine and the oscillation frequency ω are given by the relations

$$k = \left[\frac{(r_4 - r_1)(r_3 - r_2)}{(r_3 - r_1)(r_4 - r_2)}\right]^{1/2}, \quad \omega = \frac{\pi\left[\left(1 - \varkappa^2\right)(r_3 - r_1)(r_4 - r_2)\right]^{1/2}}{4K(k)} \tag{3.6}$$

where $K(k)$ is the complete elliptic integral of the first kind.

If $\varkappa > 1$, $\mu < 0$ (Fig. 1f), one oscillation in the neighbourhood $r = 0$ and one rotation correspond to each point $(\varkappa, h)$ in the region $-h_+ < h < 0$. In this region we have $r_1 < r_4 < r_3 < r_2$. On the trajectory corresponding to the oscillations ($r_1 \leqslant r \leqslant r_4$), from (3.1) we have

$$r(\tau) = \frac{r_1(r_2 - r_4) + r_2(r_4 - r_1)\operatorname{sn}^2 u}{(r_2 - r_4) + (r_4 - r_1)\operatorname{sn}^2 u}, \quad r(0) = r_1 \tag{3.7}$$

and on the trajectory corresponding to the rotation ($r_3 \leqslant r \leqslant r_2$)

$$r(\tau) = \frac{r_3(r_2 - r_4) - r_4(r_2 - r_3)\operatorname{sn}^2 u}{(r_2 - r_4) - (r_2 - r_3)\operatorname{sn}^2 u}, \quad r(0) = r_3 \tag{3.8}$$

Here too we have used notation (3.5), and the modulus k of the elliptic sine and frequency ω of oscillation or rotation have the form (3.6).

For the trajectory corresponding to rotation at $h = 0$, Eq. (3.1) is integrated in terms of elementary functions. From (3.1) and (2.2) we have

$$r(\tau)=\frac{|\mu|}{\varkappa+\cos 4\sqrt{|\mu|}\tau},\quad \varphi(\tau)=\frac{\pi}{4}+\sqrt{|\mu|}\tau$$

On trajectories of the system corresponding to rotations for $h > 0$ we have $r_3 \leqslant r \leqslant r_2$; the dependence $r(\tau)$ is given by relation (3.8), and the modulus of the elliptic function and rotation frequency are given by formulae (3.6).

If $\varkappa > 1$, $\mu > 0$ (Fig. 1d), for oscillations of the system near the origin of coordinates we have $r_1 < r_4 < r_3 \leqslant r \leqslant r_2$; the dependence $r(\tau)$ on these trajectories is given by relation (3.8), and the modulus of the elliptic function and frequency is given by formulae (3.6).

We will now consider the region of oscillations of the system in the neighbourhood of stable equilibrium positions for which $r_* = |\mu|/(2(\varkappa-1)$ (the case $\varkappa > 1$, $\mu < 0$, Fig. 1(f), $h_- < h < -h_+$). In this region, r_1 and r_2 are real, and r_3 and r_4 are complex-conjugates. On trajectories of the given region $r_1 \leqslant r \leqslant r_2$, and the dependence $r(\tau)$ is given by the relation [3]

$$r(\tau)=\frac{qr_2+pr_1-(qr_2-pr_1)\operatorname{cn}u}{q+p+(p-q)\operatorname{cn}u},\quad r(0)=r_1,\quad u=4\sqrt{2|h|}\tau \tag{3.9}$$

$$p^2=(m-r_2)^2+n^2,\quad q^2=(m-r_1)^2+n^2$$

$$m=\frac{|\mu|}{2(\varkappa+1)},\quad n=\frac{\left[-\mu^2-4h(\varkappa+1)\right]^{1/2}}{2(\varkappa+1)}$$

In (3.9) k and ω have the form

$$k=\frac{1}{2}\left[\frac{\mu^2+4h(\varkappa-1)}{2|h|}\right]^{1/2},\quad \omega=\frac{2\sqrt{2}\pi\sqrt{|h|}}{K(k)} \tag{3.10}$$

We will now give the results of integrating Eq. (3.1) on the bounded separatrices of the system. For $0 < \varkappa < 1$, $\mu > 0$ (Fig. 1c) on the separatrix joining unstable singular points, we have $r_3 \leqslant r < r_2$, where $r_3 = r_3^* = \mu(\sqrt{2}-\sqrt{(1-\varkappa)})/(2(1+\varkappa)\sqrt{(1-\varkappa)})$, $r_2 = \mu/(2(1-\varkappa))$. From (3.1) we have [3]

$$r(\tau)=\frac{\mu\left(1+e^{2u}\right)}{2(1-\varkappa)\left(1+e^{2u}\right)+4\sqrt{2(1-\varkappa)}e^{u}},\quad r(0)=r_3^*,\quad u=\frac{2\sqrt{2}\mu|\tau|}{\sqrt{1-\varkappa}} \tag{3.11}$$

For a bounded separatrix for $0 < \varkappa < 1$, $\mu < 0$ (Fig. 1e) we have $r_1 \leqslant r < r_3$. The dependence $r_1 = r_1^*, r_3 = |\mu|/(2(1+\varkappa))$ for this trajectory is obtained from the relation for $r(\tau)$ in (3.11), and the quantity r_1^* is obtained from r_3^* by replacing μ by $|\mu|$ and $\varkappa$ by $-\varkappa$.

Finally, if $\varkappa > 1$, $\mu < 0$ (Fig. 1f) we have $r_1 \leqslant r < r_3$ $(r_1 = r_1^*, r_3 = |\mu|/(2(\varkappa+1)))$ for the inner and $r_3 < r \leqslant r_2$ $(r_2 = r_2^* = |\mu|(\sqrt{(2)}+\sqrt{(\varkappa+1)})/(2(\varkappa-1)\sqrt{(\varkappa+1)}))$ for the outer separatrix. The dependence $r(\tau)$ on these is written in the form

$$r(\tau)=\frac{|\mu|\left(1+e^{2u}\right)}{2(\varkappa+1)\left(1+e^{2u}\right)\pm 4\sqrt{2(\varkappa+1)}e^{u}},\quad u=-\frac{2\sqrt{2}|\mu||\tau|}{\sqrt{\varkappa+1}}$$

where the upper and lower signs correspond to the inner separatrix (for which $r(0) = r_1^*$) and the outer separatrix (for which $r(0) = r_2^*$), respectively.

Testing for non-degeneracy. We will verify that the condition for the Hamiltonian γ_0 to be non-degenerate in regions of oscillations and rotations of the system (Fig. 1c–f) is satisfied.

We will introduce the action-angle variables I and w [4], putting

$$I(h)=\frac{1}{2\pi}\oint r(\varphi,h)d\varphi \tag{3.12}$$

The integral in (3.12) is taken along the closed trajectory $r = r(\varphi, h)$ defined by relation (2.2). The function $h = h(I)$, the inverse of (3.12), is the Hamiltonian γ_0 written in action–angle variables.

Since $d^2h/dI^2 = \omega d\omega/dh$, the condition of non-degeneracy $d^2h/dI^2 \neq 0$ reduces to the condition $d\omega/dh \neq 0$.

In the region of oscillations for $0 < \varkappa < 1$, $\mu > 0$ (Fig. 1*c*) from (3.2) and (3.6) we have

$$\frac{d\omega}{dh} = \frac{\pi\left\{K(k)\left[\varkappa\mu^2 + \varkappa s - 4h\left(1-\varkappa^2\right)\right] - 2\mu^2\, dK/dk\right\}}{2\sqrt{2}K^2(k)s\left[\mu^2 + 4\varkappa h + s\right]^{1/2}} \tag{3.13}$$

$$s = \left[\mu^4 + 8\varkappa h\mu^2 - 16h^2\left(1-\varkappa^2\right)\right]^{1/2}$$

Computer calculations showed that for each value of the parameter μ ($\mu > 0$) in the plane of the parameters ($\varkappa$ h) inside the given region of oscillations there is a curve $h = h(\varkappa, \mu)$ on which the expression in braces in (3.13) vanishes. The graph of this curve is represented qualitatively by the dashed line in Fig. 2(a). Above the curve $d\omega/dh < 0$ and below the curve $d\omega/dh > 0$.

Thus, almost everywhere in the given region of oscillations (apart from a set of zero measure) the non-degeneracy condition is satisfied.

Similarly, inside the region of oscillations for $0 < \varkappa < 1$, $\mu < 0$ (Fig. 1e) from (3.2) and (3.6) we obtain an expression which is the same as (3.13) but with $-|h|$ in the numerator instead of h and $-dK/dk$ instead of dK/dk, and inside the region of oscillations near stable equilibria for which $r_* = |\mu|/(2(\varkappa-1))$ (the case $\varkappa > 1$, $\mu < 0$, Fig. 1f) from (3.2) and (3.10) we have

$$\frac{d\omega}{dh} = -\frac{\sqrt{2}\pi}{K^2(k)}\left[\frac{K(k)}{|h|^{1/2}} + \frac{\mu^2 dK/dk}{8k|h|^{3/2}}\right]$$

Since $dK/dk > 0$, we conclude that in the first of these cases $d\omega/dh > 0$, and in the second $-d\omega/dh < 0$ for all the given values of μ, $\varkappa$, h. Thus, the non-degeneracy condition holds everywhere in both regions.

For oscillations of the system near the origin of coordinates for $\varkappa > 1$ and for rotations for $\varkappa > 1$, $\mu < 0$ (Fig. 1d, f) it is easy to test the non-degeneracy condition using the relation

$$\frac{d^2h}{dI^2} = \frac{\omega^3}{2\pi}\int \frac{\partial^2\gamma_0/\partial r^2}{(\partial\gamma_0/\partial r)^3}d\varphi = \frac{\omega^3}{2\pi}\int \frac{2(\kappa - \cos 4\varphi)}{(\partial\gamma_0/\partial r)^3}d\varphi \tag{3.14}$$

obtained from (2.2) and (3.12).

The numerator of the fraction in the integrand in (3.14) is positive for $\varkappa > 1$. For trajectories corresponding to oscillations near the origin of coordinates for $\varkappa > 1$, $\mu < 0$ (Fig. 1f), $d\varphi < 0$ and the denominator of the fraction in (3.14) is negative ($\partial\gamma_0/\partial r = d\varphi/\partial\tau < 0$), since the angle φ is monotonely decreasing on those trajectories. Thus, $d^2h/dI^2 > 0$, and the non-degeneracy condition holds. For trajectories corresponding to rotations of the system for $\varkappa > 1$, $\mu < 0$ (Fig. 1f) or oscillations in the neighbourhood of the origin for $\varkappa > 1$, $\mu > 0$ (Fig. 1d) $d\varphi > 0$ and the denominator of the fraction in (3.14) is positive (the angle φ increases monotonely). Hence, $d^2h/dI^2 > 0$ and the non-degeneracy condition holds.

4. NON-LINEAR OSCILLATIONS OF THE COMPLETE SYSTEM

We will now show how the results for the model system described by the Hamiltonian γ_0 can be applied to a complete system with Hamiltonian Γ (cf. 1.3)).

According to Poincaré's theory of periodic motion [5], for sufficiently small values of ε a unique solution of the complete system with is $8\pi\varepsilon^2$-periodic in τ and analytic with respect to ε is generated from each position of equilibrium of the model system which does not coincide with the origin of coordinates, corresponding to motion of the initial system which is 8π-periodic with respect to τ with Hamiltonian (1.1).

The unstable equilibrium positions of system (2.1) correspond to unstable but periodic solutions of the complete system: this follows from the fact that the characteristic exponents of the corresponding linear equations of perturbed motion are continuous with respect to ε.

We will use the Arnol'd-Moser theorem [6, 7] to examine the stability of the periodic motions generated from stable equilibrium positions ($\varkappa > 1$, $\mu < 0$, Fig. 1f) of the model system. To obtain the normal form of the Hamiltonian function of perturbed motion, we first normalize the Hamiltonian γ_0 of the model system in the neighbourhood of these equilibria. Putting $\varphi = \varphi_* + \xi$, $r = r_* + \eta$ ($\varphi_* = 0$, $\pi/2$, π, $3\pi/2$; $r_* = |\mu|/(2(\varkappa-1)))$, we can represent γ_0 in the form

$$\gamma_0 = \gamma_0^{(2)} + \gamma_0^{(3)} + \gamma_0^{(4)} + \dots \tag{4.1}$$

$$\gamma_0^{(2)} = 8r_*^2\xi^2 + (\varkappa-1)\eta^2,\quad \gamma_0^{(3)} = 16r_*\xi^2\eta,\quad \gamma_0^{(4)} = 8\xi^2\eta^2 - (32/3)r_*^2\xi^4$$

where the dots denote the set of terms of higher than the fourth power in ξ and η. The replacement of variables $\xi = \xi_1/\alpha$, $\eta = \alpha\eta_1$, $\alpha = 2^{3/4}\sqrt{(r_*)}(\varkappa-1)^{-1/4}$ reduces the quadratic part $\gamma_0^{(2)}$ to normal form $\omega(\xi_1^2 + \eta_1^2)/2$, $\omega = 2\sqrt{(2)}|\mu|(\varkappa-1)^{-1/2}$. We then make a canonical Birkhoff transformation $\xi_1, \eta_1 \to q, p$ which removes cubic terms in the Hamiltonian and simplifies terms of the fourth power. In the neighbourhood of the given equilibrium, the normal form of the Hamiltonian γ_0 will be

$$Z = \frac{1}{2}\omega(q^2+p^2) + \frac{1}{4}c(q^2+p^2)^2 + \dots,\quad c = -2(\varkappa+3) \tag{4.2}$$

We will now normalize the complete Hamiltonian Γ in the neighbourhood of the periodic solution of the complete system generated by the given stable equilibrium. The normalized Hamiltonian will have the form (4.2), where corrections of order ε are made to the coefficients ω and c. For sufficiently small values of ε, by virtue of the inequality $c < 0$, the Hamiltonian Γ is non-degenerate in the neighbourhood of the periodic solution considered above. Hence, by the Arnol'd–Moser theorem, it is Lyapunov stable.

We will now show that motions of the complete system starting in a finite neighbourhood of the origin of coordinates are bounded, and estimate the size of that neighbourhood.

At the end of Section 3 we showed that for all closed trajectories of the model system (2.1) (corresponding to oscillations and rotations) the condition for non-degeneracy of the Hamiltonian γ_0 is satisfied. In particular, for cases $0 < \varkappa < 1$, $\mu > 0$ (Fig. 1c) and $0 < \varkappa < 1$, $\mu < 0$ (Fig. 1e) this condition is satisfied, for example, by the trajectory corresponding to an oscillation in the neighbourhood of the origin of coordinates on which the maximum value of r is not greater than the value $r_3^*/2$ and $r_1^*/2$, respectively, where r_3^* and r_1^* are the minimum values of r on the corresponding separatrix defined at the beginning of Section 3.

According to Moser's theorem concerning invariant curves [7], for sufficiently small values of ε the mapping generated by motions of the complete system after period $8\pi\varepsilon^2$ has an invariant curve close to the given trajectory. For all trajectories of the complete system which start inside this curve for cases $0 < \varkappa < 1$, $\mu > 0$ and $0 < \varkappa < 1$, $\mu < 0$ respectively, we have

$$r(\tau) < \frac{1}{2}r_3^*(1+O(\varepsilon)) \quad\text{and}\quad r(\tau) < \frac{1}{2}r_1^*(1+O(\varepsilon))$$

If $\varkappa > 1$, $\mu < 0$ (Fig. 1f), then, as in Moser's theorem, we choose one of the invariant curves, close to a trajectory corresponding to rotation, say, on which the value of r is no greater than $2r_2^*$, where r_2^* is the maximum value of r on the outer separatrix (see Section 3). For all trajectories of the complete system which start inside the given invariant curve we have

$$|r(\tau)| < 2r_2^*(1+O(\varepsilon)) \tag{4.3}$$

Finally, for the case $\varkappa > 1$, $\mu > 0$ (Fig. 1d) we can estimate the size of the neighbourhood of the origin of coordinates beyond which trajectories which start inside that neighbourhood will not go, using, say, the inequality (4.3).

5. EXAMPLES

We will give two examples to illustrate the results.

1. Consider the motion of a point mass of mass m above a fixed absolutely smooth curve in the shape of an ellipse, given in a fixed system of coordinates $O\xi\eta$ by the equation $\xi^2a^{-2} + (\eta-b)^2b^{-2} = 1$ (the $O\eta$ axis is vertical). Moving in the plane $O\xi\eta$, the point occasionally collides with the curve; the collision is assumed to be completely elastic and frictionless.

There is a periodic motion of the point when its trajectory lies on the $O\eta$ axis, and as a result of colliding with an arc of the ellipse the point at the origin of coordinates $\xi = \eta = 0$ periodically jumps a height l; the period of this motion is equal to $2\sqrt{(2l/g)}$. The isoenergetic orbital stability of this motion has been investigated in [8].

Suppose that in the perturbed motion immediately before the first and second collisions of the point we have $\xi = lx$, $p_\xi = m\sqrt{(2gl)}y$ and $\xi = lx_1$, $p_\xi = m\sqrt{(2gl)}y_1$, respectively (ξ and p_ξ are the generalized coordinate and momentum). In [8] the investigation of the isoenergetic orbital stability of the given periodic motion of a point was reduced to the investigation of the stability of a fixed point $x = y = 0$ of the area-preserving mapping

$$x_1 = x_1(x, y), \quad y_1 = y_1(x, y) \tag{5.1}$$

of the plane into itself.

We showed, in particular, that on the straight line $\beta = \alpha/4$ in the plane of parameters α and β ($\alpha = a^2/b^2$, $\beta = l/b$) (Fig. 3) there is fourth-order resonance; also, if $0 < \alpha < 5$ or $\alpha > 10$ (which corresponds to the case $\varkappa > 1$ here), there is orbital stability, and for $5 < \alpha < 10$ (when $0 \leqslant \varkappa < 1$) the given periodic motion of the point is unstable [8]. At points c (5, 5/2) and d(10, 5/2)—the boundaries of the areas of stability and instability (corresponding to the critical value $\varkappa = 1$)—the vertical periodic motion of a point is respectively orbitally stable and unstable [2].

Suppose now that the value of β/α is close to $^1/_4$. Assuming $\beta/\alpha = 1/4 + \pi\varepsilon^2\chi/2$, we make the replacement of variables $x, y \to q_*, p_*$, which brings the mapping (5.1) to normal form [8]; then, changing to variables $q = \varepsilon^{-1}q_*$, $p = \varepsilon^{-1}p_*$, we obtain this normalized mapping in the form

$$\begin{aligned}
q_1 &= \cos 2\pi\lambda q + \sin 2\pi\lambda p + \varepsilon^2[\mu_{21}^* q(q^2+p^2) + \mu_{03}^* q(q^2 - 3p^2)] + O(\varepsilon^3) \\
p_1 &= -\sin 2\pi\lambda q + \cos 2\pi\lambda p + \varepsilon^2[\mu_{21}^* p(q^2+p^2) + \mu_{03}^* p(p^2 - 3q^2)] + O(\varepsilon^3) \\
\lambda &= \frac{1}{2\pi}\arccos\left(1 - \frac{4\beta}{\alpha}\right), \quad \mu_{21}^* = \frac{5}{8} - \frac{3\beta}{8}, \quad \mu_{03}^* = -\frac{\beta}{8}
\end{aligned} \tag{5.2}$$

The Hamilton function generating the mapping (5.2) after period 2π, can be written in "polar" coordinates θ, R($q = \sqrt{(2R)}\sin\theta, p = \sqrt{(2R)}\cos\theta$) as

$$H = \lambda R - \frac{\varepsilon^2}{2\pi}[\mu_{21}^* + \mu_{03}^* \cos(4\theta - t)]R^2 + O(\varepsilon^3) \tag{5.3}$$

We make the replacement of variables $\theta, R \to \varphi, r$ described in Section 1, putting $\theta = t/4 + \sigma[(1+\sigma)\pi/8 + \varphi]$, $R = 2\pi/|\mu^*_{03}|$, $\sigma = -\text{sign}\,\mu_{21}$, and introduce the new independent variable $\tau = \varepsilon^2 t$. The Hamiltonian (5.3) will take the form (1.3), (1.4), where $\mu = -\text{sign}\,\mu^*_{21\chi}$, $\varkappa = |\mu^*_{21}|/|\mu^*_{03}|$.

We will apply the results of Section 4 to the problem here.

Apart from periodic motion of a point along the vertical $O\eta$ (orbitally stable at $\mu \neq 0$), in this case there are still more periodic motions (with period $T = 8\sqrt{(2l)/g}$), in the neighbourhood of the given vertical motion. They correspond to 8π-periodic motions of a system with Hamiltonian (5.3). As the analysis shows, T-periodic motions of a point can be of two types. Neglecting terms $O(\varepsilon^2)$, the corresponding trajectories of the point have the form shown in Fig. 4. For motion of the first kind (Fig. 4a) the abscissae of the points of intersection of the trajectory of the point with the arc of the ellipse are equal to $-A, 0, A$, and for motion of the second kind, $-A\sqrt{(2)}/2, A\sqrt{(2)}/2$, where $A = 4\varepsilon l\sqrt{(\pi r_*/|\mu^*_{03}|)}$, and r_* is the corresponding equilibrium value of the variable r of the model system.

For points (α, β) lying in the neighbourhood of the straight line $\beta = \alpha/4$ (Fig. 3) in intervals $0 < \alpha < 5$ and $\alpha > 10$ (where $\varkappa > 1$), the only periodic motion of the point below that line ($\mu > 0$) is vertical; above that line ($\mu < 0$), in the same ranges of variation of α, there are another two T-periodic motions of the point—motions of the first and second types (Fig. 4). When $0 < \alpha < 5$ the motion of the first type is unstable, and that of the second type is orbitally stable; for $\alpha > 10$, on the other hand, the motion of the first and second types is orbitally stable and unstable, respectively.

In a small neighbourhood, above and below, the straight line $\beta = \alpha/4$ in the interval $5 < \alpha < 10$ (where $0 \leqslant \varkappa < 1$) ($\mu < 0$ and $\mu > 0$, respectively), in addition to vertical periodic motion of the point, there is another T-periodic motion which is unstable. For points (α, β) lying above the straight line $\beta = \alpha/4$ for $5 < \alpha < 20/3$ and below this line for $20/3 < \alpha < 10$, this motion is of the first type, but for points (α, β) below the line $\beta = \alpha/4$ for

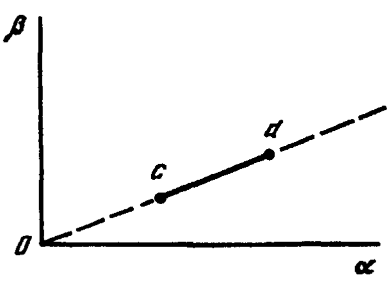

Fig. 3.

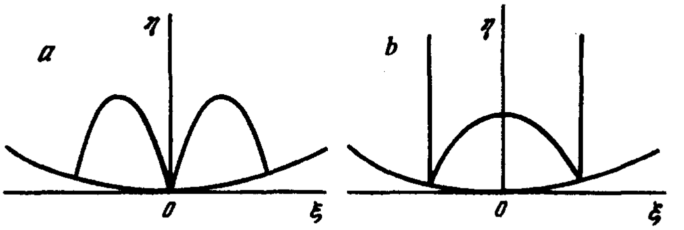

Fig. 4.

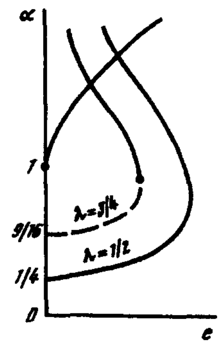

Fig. 5.

$5 < \alpha < 20/3$ and above that line for $20/3 < \alpha < 10$ it is of the second type. We have excluded from consideration the neighbourhood of the critical point (20/3, 5/3) of the straight line $\beta = \alpha/4$ (where $\varkappa = 0$).

Any motions of a point which start in a sufficiently small neighbourhood of its vertical periodic motion remain bounded: it follows from the results of Section 4 that for these motions

$$|\xi(t)| < 2l\sqrt{\rho^*}\,\varepsilon(1+O(\varepsilon))$$

where $\rho^* = r_1^*$ for the region $5 < \alpha < 10$, $\beta > \alpha/4$; $\rho^* = r_3^*$ for the region $5 < \alpha < 10$, $\beta > \alpha/4$; $\rho^* = 4r_2^*$ in regions $0 < \alpha < 5$ and $\alpha > 10$ on both sides of the line $\beta = \alpha/4$. The quantities r_i^* ($i = 1, 2, 3$) were defined in the first part of Section 3.

2. Consider plane non-linear oscillations of a satellite—a rigid body—about the centre of mass in an elliptical orbit described by the equation [9]

$$(1+e\cos\nu)\frac{d^2\psi}{d\nu^2} - 2e\sin\nu\frac{d\psi}{d\nu} + \alpha\sin\psi\cos\psi = 2e\sin\nu \tag{5.4}$$

where ψ is the angle between one of the principal central axes of the ellipsoid of inertia of the satellite, lying in the plane of the orbit, and the radius vector of its centre of mass, ν is the true anomaly, e is the eccentricity of the orbit and α is the inertial parameter ($|\alpha| \leqslant 3$).

We will consider one of the regions in the plane of parameters e, α (Fig. 5), where there is a unique 2π-periodic solution $\psi = \psi_*(\nu)$ of Eq. (5.1) which is stable in the linear approximation (the characteristic exponents $\pm i\lambda$ of the corresponding linearized system are pure imaginary) and which transfers when $e = 0$ to the equilibrium position of a satellite in the orbital system of coordinates [10]. This region is bounded by the branching curve (issuing from the point (0, 1)), the curve $\lambda = 1/2$ and the straight line $e = 0$ (Fig. 5).

There is fourth-order resonance on the curve $\lambda = 3/4$ which passes through the given region. On the part of the resonance curve depicted by the dashed line in Fig. 5, the solution $\psi_*(\nu)$ is stable (here $\varkappa > 1$), while on the other part, depicted by the solid line, it is unstable ($0 < \varkappa < 1$); the boundary point of the regions of stability and instability $\varkappa = 1$ has coordinates (0.097; 0.85) [11].

We select part of the resonance curve near the boundary point on either side of this point and consider the existence and stability of 8π-periodic motions of the satellite for values of a and α lying in a small neighbourhood of the given part of the curve $\lambda = 3/4$. Everywhere in this neighbourhood the coefficient c of the normal form of the Hamilton function is negative [11]; for points (e, α) to the left and right of the resonance curve we have $\chi > 0$, $\mu < 0$ and $\chi < 0$, $\mu > 0$, respectively (the notation for c, χ and μ was introduced in Section 1). Thus (cf. Section 4), for points (e, α) which are on either side of the resonance curve above its boundary point (where $0 < \varkappa < 1$), there is unstable 8π-periodic motion of the satellite in the neighbourhood of its 2π-periodic motion, described by the solution $\psi = \psi_*(\nu)$. For points (e, α) in the neighbourhood of the resonance curve where $\varkappa > 1$ (below the boundary point), there are two 8π-periodic motions to the left of this curve, one of which is stable and the other not; for points (e, α) to the right of this part of the curve, there are no 8π-periodic motions of the satellite different from the given 2π-periodic motion.

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